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2008 J. Phys. A: Math. Theor. 41 415401

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The boundary conditions for point transformed electromagnetic invisibility cloaks*

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Received 27 March 2008

Published 15 September 2008

Online at stacks.iop.org/JPhysA/41/415401

Abstract

In this paper we study point transformed electromagnetic invisibility cloaks in transformation media that are obtained by transformation from general anisotropic media. We assume that there are several point transformed electromagnetic cloaks located in different points in space. Our results apply in particular to the first-order invisibility cloaks introduced by Pendry *et al* and to the high-order invisibility cloaks introduced by Hendi *et al* and by Cai *et al*. We identify the appropriate *cloaking boundary conditions* that the solutions of Maxwell equations have to satisfy at the outside, ∂K_+ , and at the inside, ∂K_- , of the boundary of the cloaked object K in the case where the permittivity and the permeability are bounded below and above in K . Namely, that the tangential components of the electric and the magnetic fields have to vanish at ∂K_+ —which is always true—and that the normal components of the curl of the electric and the magnetic fields have to vanish at ∂K_- . These results are proven requiring that energy be conserved. In the case of one spherical cloak with a spherically stratified K and a radial current at ∂K we verify by an explicit calculation that our *cloaking boundary conditions* are satisfied and that cloaking of active devices holds, even if the current is at the boundary of the cloaked object. As we prove our results for media that are obtained by transformation from general anisotropic media, our results apply to the cloaking of objects with passive and active devices contained in general anisotropic media, in particular to objects with passive and active devices contained inside general crystals. Our results suggest a method to enhance cloaking in the approximate transformation media that are used in practice. Namely, to coat the boundary of the cloaked object (the inner boundary of the cloak) with a material that imposes the boundary conditions above. As these boundary conditions have to be satisfied for exact transformation media, adding a lining that enforces them

* Research partially supported by Conacyt under project P42553F.

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in the case of approximate transformation media will improve the performance of approximate cloaks.

PACS numbers: 41.20.Jb, 02.30.Tb, 02.30.Zz, 02.60.Lj

Mathematics Subject Classification: 35L45, 35L50, 35L80, 35P25, 35Q60, 78A25, 78A45

1. Introduction

In this paper we study point transformed electromagnetic invisibility cloaks in transformation media that are obtained by transformation from general anisotropic media. We assume that there are several cloaks located in different points in space. Our results apply in particular to the first-order invisibility cloaks introduced by [1] and to the high-order invisibility cloaks introduced by [2, 3].

In [4, 5] we gave a rigorous proof—based on energy conservation—of cloaking of passive and active devices for our general class of invisibility cloaks. The cloaked object, K , completely decouples from the exterior. Actually, the cloaking outside is independent of what is inside K . The electromagnetic waves inside K cannot leave K and vice versa, the electromagnetic waves outside cannot go inside. Furthermore, we identified the appropriate *cloaking boundary conditions* when cloaking is formulated as a boundary value problem. We proved that the tangential components of the electric and the magnetic fields have to vanish at the outside of the boundary of the cloaked object, ∂K_+ . This boundary condition is self-adjoint in our case because the permittivity and the permeability are degenerate at ∂K_+ . We also proved that the boundary condition at the inside of the boundary of the cloaked object, ∂K_- , can be any self-adjoint boundary condition for the Maxwell generator in K . This is true in the general case where the permittivity and the permeability are allowed to be degenerate at ∂K_- . In this general situation the particular boundary condition that nature will take on ∂K_- will depend on the behavior of the permittivity and the permeability near ∂K_- . We proved our results both in the time and in the frequency domains, and as we consider media obtained by transformation from general anisotropic media, our results apply, in particular, to objects inside general crystals.

In this paper, we address the problem of determining the *cloaking boundary conditions* at ∂K_- in the case where the permittivity and the permeability inside K are bounded and have a positive lower bound. This corresponds to the situation where we have a standard object K —that could be anisotropic and inhomogeneous, but whose permittivity and permeability are neither singular nor degenerate—that is coated by a transformation medium that is degenerate at ∂K_+ . We also allow for active devices in K . This is perhaps the more important case in the applications. We prove that in this case the *cloaking boundary conditions* at ∂K_- are that the normal components of the curl of the electric and the magnetic fields vanish.

Since we have identified the *cloaking boundary conditions* at ∂K_{\pm} we have now a complete formulation of cloaking as a boundary value problem in this important case. For the exact transformation media that we consider in this paper these boundary conditions are satisfied because they follow from energy conservation and there is no need to add any lining to impose on them. In other words, they are the conditions taken by nature, as they are imposed by energy conservation. However, our results suggest a method to enhance cloaking in the approximate transformation media that are used in practice. Namely, to coat the boundary of the cloaked object (the inner boundary of the cloak) by a material that imposes the boundary

conditions above. As these boundary conditions have to be satisfied for exact transformation media, adding a lining that enforces them in the case of approximate transformation media will improve the performance of approximate cloaks.

It is, of course, a well-known fact in electromagnetic theory—and in wave propagation in general—that in any interphase between two different media there has to be a boundary condition. This obviously applies to the interphase between the cloaked object and the coating metamaterial, i.e., ∂K . So, the real question is not if there has to be boundary conditions at ∂K_{\pm} , but rather what are the appropriate boundary conditions. We address this question in this paper as well as in [4, 5]. The reason why this is a delicate problem that requires a careful analysis is that for point transformed electromagnetic cloaks the permittivity and the permeability are degenerate at ∂K and in consequence the standard rules that are used in the non-degenerate case do not apply. In fact, the solutions to Maxwell equations are, in general, discontinuous at ∂K .

The interesting paper [6] considers point and line transformed electromagnetic cloaks under general coordinate transformations. Among other problems, they compute the fields outside the cloaked object from the fields in the original electromagnetic space, using the transformation formulae between them, what avoids doing tedious calculations in the transformed space. This method was previously used in [4, 5] for spherical and cylindrical cloaks. In this way, it is proven in [6] that for general point transformed cloaks the tangential components of the electric and the magnetic fields vanish at the outside of the boundary of the cloaked object and also that for general line transformed cloaks the tangential components of the electric and magnetic fields that are orthogonal to the axis of the cloak vanish at the outside of the boundary of the cloaked object. This generalizes the results previously proved in [4, 5] in the case of spherical and cylindrical cloaks, using the same method.

The paper [7] considers cloaking in terms of the Cauchy data, in the context of the Dirichlet to Neumann operator. Among other problems, they study cloaking of passive and active devices for one spherical electromagnetic cloak. They postulate a class of weak solutions in distribution sense across ∂K (see definition 4 of [7]). They study the case when the permittivity and permeability are bounded below and above inside K (what they call the single coating) in theorem 5, where, among other results, they prove that the tangential components of the electric and magnetic fields of their solutions have to vanish at ∂K_{-} . They conclude that their solutions do not exist for generic currents inside K , and that cloaking holds for passive devices but that it fails for active devices with generic currents inside K . To deal with this issue they propose to add a perfect electric conducting lining to K —what makes it to appear as passive—or to introduce a different construction that they call the double coating.

In this paper, as well as in [4, 5], we proceed in a completely different way. Instead of postulating *a priori* a particular class of weak solutions in distributions sense across ∂K we first characterize all possible ways to define solutions that are compatible with energy conservation. They correspond to all self-adjoint extensions of the Maxwell generator. Note that each self-adjoint extension can be understood in terms of boundary conditions at ∂K_{\pm} . We proved in [4, 5] that all self-adjoint extensions are the direct sum of some self-adjoint extension inside K with a fixed self-adjoint extension outside K . This implies that the solutions inside and outside of K are completely decoupled from each other and that, in general, they are discontinuous at ∂K . Another consequence is that in the case where the permittivity and the permeability are bounded below and above inside K weak solutions in distribution sense across ∂K , and more generally solutions with transmission conditions that link the inside and the outside of K , are not self-adjoint, i.e., they are not allowed by energy conservation. Note that in this case, as the permittivity and the permeability are bounded below and above inside K , requiring that the tangential components of both, the electric and the magnetic field

vanish at ∂K_- is not a self-adjoint boundary condition. We are only allowed to require that one of them vanishes. However, requiring that both vanish at ∂K_+ is a self-adjoint boundary condition because the permittivity and the permeability are degenerate at ∂K_+ .

We also proved in [4, 5] that cloaking of passive and active devices always holds for all possible ways to define solutions that satisfy energy conservation, i.e., with self-adjoint boundary conditions.

There are many papers that discuss line transformed, or cylindrical, cloaks (see, for example, [1, 3, 4, 6, 7, 10–12, 14, 16, 18–20]). In [18] boundary conditions are considered to enhance cloaking for a cylindrical cloak. Note that the results for line transformed—or cylindrical—cloaks are quite different from the ones for point transformed cloaks studied in this paper.

As it is often the case in the papers on electromagnetic invisibility cloaks, we make the assumption that the media are not dispersive. This is a widely used idealization. As is well known, metamaterials are dispersive, and, furthermore, when the permittivity and the permeability have eigenvalues less than one, dispersion comes into play in order that the group velocity does not exceeds the speed of light. This idealization means that we have to take a narrow enough range of frequencies in order that we can analyze the cloaking effect without taking dispersion into account. In practice this means that cloaking will only be approximate. Note, moreover, that the results in this paper, as well as in [4, 5], are proven for cloaks in exact transformation media.

The paper is organized as follows. In section 2, we prove that our *cloaking boundary conditions* at ∂K_- are satisfied. In section 3, we illustrate our method by considering one spherical cloak with an active device given by a radial electric current at the boundary of K . The case of a magnetic current at the boundary of K follows in the same way. We assume that K is isotropic and spherically stratified. We verify in this particular case, by an explicit computation, that our *cloaking boundary conditions* are satisfied and that cloaking of active devices holds, even if the current is at the boundary of the cloaked object, as we have proven in section 2 in the general case where there is no explicit solution. We end the paper with conclusions.

For other results on invisibility cloaks see [13, 15, 17] as well as the references quoted there and in [4, 5].

2. The boundary conditions

Let us consider Maxwell equations in \mathbb{R}^3 , in the time domain,

$$\nabla \times \mathbf{E} = -\frac{\partial}{\partial t} \mathbf{B}, \quad \nabla \times \mathbf{H} = \frac{\partial}{\partial t} \mathbf{D}, \quad (2.1)$$

$$\nabla \cdot \mathbf{B} = 0, \quad \nabla \cdot \mathbf{D} = 0, \quad (2.2)$$

and in the frequency domain, assuming a periodic time dependence of \mathbf{E} , \mathbf{H} given by $e^{i\omega t}$, with ω the frequency,

$$\nabla \times \mathbf{E} = -i\omega \mathbf{B}, \quad \nabla \times \mathbf{H} = i\omega \mathbf{D}, \quad \omega \neq 0, \quad (2.3)$$

$$\nabla \cdot \mathbf{B} = 0, \quad \nabla \cdot \mathbf{D} = 0, \quad (2.4)$$

where we have suppressed the factor $e^{i\omega t}$ in both sides. In this paper we take the time factor $e^{i\omega t}$ to use the convention of [8, 9]. Note that (2.4) follows from (2.3).

We briefly recall some notations and definitions from [5].

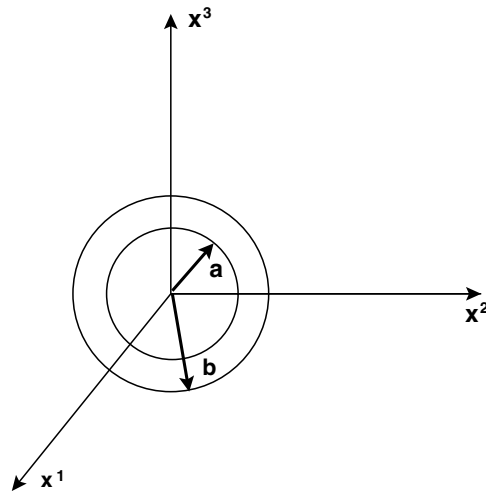


Figure 1. One spherical cloak centered at zero.

Let us first consider the case where there is only one cloak located at $\mathbf{x} = 0$ (see figure 1). We designate the Cartesian coordinates of \mathbf{x} by x^λ , $\lambda = 1, 2, 3$. To define the transformation media we introduce another copy of \mathbb{R}^3 , denoted by \mathbb{R}_0^3 . The points in \mathbb{R}_0^3 are denoted by \mathbf{y} with coordinates y^λ , $\lambda = 1, 2, 3$. We designate $\hat{\mathbf{x}} := \mathbf{x}/|\mathbf{x}|$, $\hat{\mathbf{y}} := \mathbf{y}/|\mathbf{y}|$. Consider the following transformation from $\mathbb{R}_0^3 \setminus \{0\}$ to \mathbb{R}^3 :

$$\mathbf{x} = \mathbf{x}(\mathbf{y}) = f(\mathbf{y}) := g(|\mathbf{y}|)\hat{\mathbf{y}}. \tag{2.5}$$

In spherical coordinates this transformation changes the radial coordinate but leaves the angular coordinates constant, i.e., $|\mathbf{x}| = g(|\mathbf{y}|)$, $\hat{\mathbf{x}} = \hat{\mathbf{y}}$. Given $0 < a < b$ we wish that this transformation sends the punctuated ball $0 < |\mathbf{y}| \leq b$ onto the concentric shell $a < |\mathbf{x}| \leq b$, that it is the identity for $|\mathbf{y}| \geq b$ and that it is one to one. Then, we assume that g satisfies the following conditions.

Definition 2.1. For any positive numbers a, b with $0 < a < b$, we say the g is a cloaking function in $[0, b]$ if $g(\rho)$ is twice continuously differentiable on $[0, b]$, $g(0) = a$, $g(b) = b$, and $g'(\rho) := \frac{d}{d\rho}g(\rho) > 0$, $\rho \in [0, b]$.

We define

$$\begin{aligned} \mathbf{x} = \mathbf{x}(\mathbf{y}) = f(\mathbf{y}) &:= g(|\mathbf{y}|)\hat{\mathbf{y}}, & \text{for } 0 < |\mathbf{y}| \leq b, \\ \mathbf{x} = \mathbf{x}(\mathbf{y}) &:= \mathbf{y}, & \text{for } |\mathbf{y}| \geq b. \end{aligned} \tag{2.6}$$

With these conditions (2.6) is a bijection from $\mathbb{R}_0^3 \setminus \{0\}$ onto $\mathbb{R}^3 \setminus B_a(0)$, where we denote

$$B_r(\mathbf{x}_0) := \{\mathbf{x} \in \mathbb{R}^3 : |\mathbf{x} - \mathbf{x}_0| \leq r\}. \tag{2.7}$$

Moreover, it blows up the point 0 onto the sphere $|\mathbf{x}| = a$. It sends the punctuated ball $0 < |\mathbf{y}| \leq b$ onto the concentric shell $a < |\mathbf{x}| \leq b$ and it is the identity for $|\mathbf{y}| \geq b$. It is twice continuously differentiable away from the sphere $|\mathbf{y}| = b$, where it can have discontinuities in the derivatives depending on the values of the derivatives of g at b .

In [3] the quadratic case

$$g(\rho) = \left[1 - \frac{a}{b} + p(\rho - b)\right] \rho + a$$

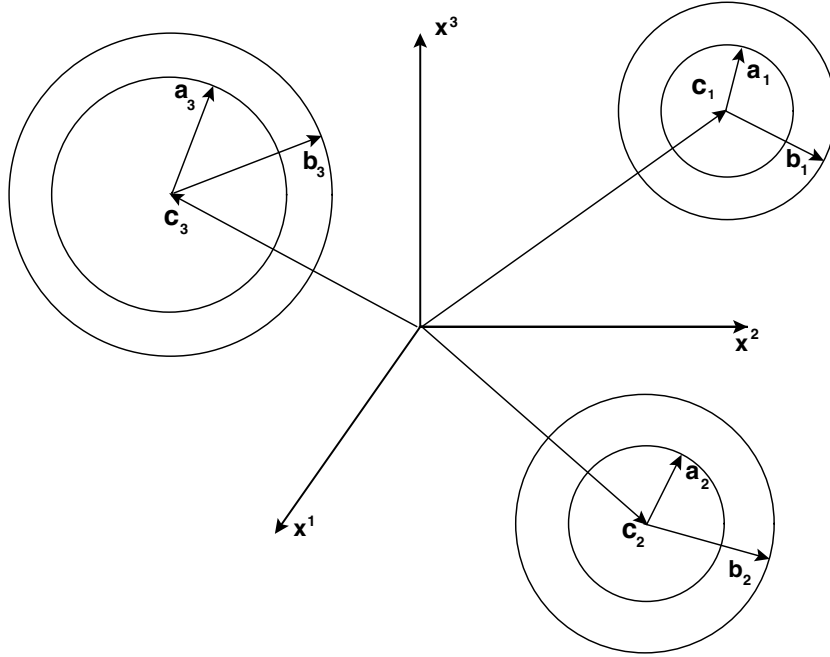


Figure 2. Three spherical cloaks centered at $\mathbf{c}_1, \mathbf{c}_2, \mathbf{c}_3$.

with $p \in \mathbb{R}$ was discussed in connection with a cylindrical cloak in an approximate transformation medium. In [1] the first-order case $g(\rho) = \frac{b-a}{b}\rho + a$ was considered. First-order transformations were previously used in [21, 22] in the context of Calderón’s inverse conductivity problem.

The closed ball $K := \{\mathbf{x} \in \mathbb{R}^3 : |\mathbf{x}| \leq a\}$ is the region that we wish to conceal, and we call it the cloaked object. The spherical shell $a < |\mathbf{x}| \leq b$ is the cloaking layer. The union of the cloaked object and the cloaking layer is the spherical cloak. The domain $|\mathbf{x}| > b$ is the exterior of the spherical cloak.

We now put a finite number of spherical cloaks in different points in space in such a way that they do not intersect (see figure 2). Let us take as centers of the cloak points $\mathbf{c}_j \in \mathbb{R}^3, j = 1, 2, \dots, N$, where N is the number of cloaks and $\mathbf{c}_j \neq \mathbf{c}_l, j \neq l, 1 \leq j, l \leq N$. We take $0 < a_j < b_j$, and cloaking functions g_j that satisfy the conditions of definition 1 for $a_j, b_j, j = 1, 2, 3, \dots, N$, and we define the following transformation from $\mathbb{R}_0^3 \setminus \{\mathbf{c}_1, \mathbf{c}_2, \dots, \mathbf{c}_N\}$ to \mathbb{R}^3 :

$$\begin{aligned} \mathbf{x} = \mathbf{x}(\mathbf{y}) = f(\mathbf{y}) &:= \mathbf{c}_j + g_j(|\mathbf{y} - \mathbf{c}_j|)\widehat{\mathbf{y} - \mathbf{c}_j}, & \mathbf{y} \in B_{b_j}(\mathbf{c}_j), & \quad j = 1, 2, \dots, N, \\ \mathbf{x} = \mathbf{x}(\mathbf{y}) = f(\mathbf{y}) &:= \mathbf{y}, & \mathbf{y} \in \mathbb{R}_0^3 \setminus \bigcup_{j=1}^N B_{b_j}(\mathbf{c}_j), \end{aligned} \tag{2.8}$$

where $B_{b_j}(\mathbf{c}_j)$ are balls in \mathbb{R}_0^3 .

The cloaked objects that we wish to conceal are given by

$$K_j := \{\mathbf{x} \in \mathbb{R}^3 : |\mathbf{x} - \mathbf{c}_j| \leq a_j\}, \quad j = 1, 2, \dots, N. \tag{2.9}$$

The spherical shells $a_j < |\mathbf{x} - \mathbf{c}_j| \leq b_j, j = 1, 2, \dots, N$ are the cloaking layers. The spherical cloaks are the balls $B_{b_j}(\mathbf{c}_j)$ in \mathbb{R}^3 . We denote by K the union of all the cloaked objects,

$$K := \bigcup_{j=1}^N K_j. \tag{2.10}$$

The domain

$$\mathbb{R}^3 \setminus \bigcup_{j=1}^N B_{b_j}(\mathbf{c}_j) \quad (2.11)$$

is the exterior of all the spherical cloaks. We assume that the spherical cloaks are at a positive distance of each other,

$$\min \text{distance} (B_{b_j}(\mathbf{c}_j), B_{b_l}(\mathbf{c}_l)) > 0, \quad j \neq l, \quad j, l = 1, 2, \dots, N.$$

Denote

$$\Omega_0 := \mathbb{R}_0^3 \setminus \{\mathbf{c}_1, \mathbf{c}_2, \dots, \mathbf{c}_N\}, \quad \Omega := \mathbb{R}^3 \setminus K.$$

Then, (2.8) is a bijection from Ω_0 onto Ω , and for $j = 1, 2, \dots, N$ it blows up the point \mathbf{c}_j onto the sphere $|\mathbf{x} - \mathbf{c}_j| = a_j$. It sends the punctuated ball $0 < |\mathbf{y} - \mathbf{c}_j| \leq b_j$ onto the shell $a_j < |\mathbf{x} - \mathbf{c}_j| \leq b_j$ and it is the identity for $\mathbf{y} \in \mathbb{R}_0^3 \setminus \text{interior}(\bigcup_{j=1}^N B_{b_j}(\mathbf{c}_j))$. It is twice continuously differentiable away from the spheres $|\mathbf{y} - \mathbf{c}_j| = b_j$, where it can have discontinuities in the derivatives depending on the values of the derivatives of g_j at b_j .

The elements of the Jacobian matrix are denoted by $A_{\lambda'}^{\lambda}$,

$$A_{\lambda'}^{\lambda} := \frac{\partial x^{\lambda}}{\partial y^{\lambda'}}. \quad (2.12)$$

$A_{\lambda'}^{\lambda} \in C^1(\Omega_0 \setminus \bigcup_{j=1}^N \partial B_{b_j}(\mathbf{c}_j))$, and that it can have discontinuities on $\bigcup_{j=1}^N \partial B_{b_j}(\mathbf{c}_j)$ depending on the derivatives of g_j at b_j . We designate by $A_{\lambda}^{\lambda'}$ the elements of the Jacobian of the inverse bijection, $\mathbf{y} = \mathbf{y}(\mathbf{x}) = f^{-1}(\mathbf{x})$,

$$A_{\lambda}^{\lambda'} := \frac{\partial y^{\lambda'}}{\partial x^{\lambda}}. \quad (2.13)$$

$A_{\lambda}^{\lambda'} \in C^1(\Omega \setminus \bigcup_{j=1}^N \partial B_{b_j}(\mathbf{c}_j))$, and it can have discontinuities on $\bigcup_{j=1}^N \partial B_{b_j}(\mathbf{c}_j)$ depending on the derivatives of g_j at b_j .

It follows from (2.8) that the transformation matrix (2.12) is given by

$$A_{\lambda'}^{\lambda} = \frac{g_j(|\mathbf{y} - \mathbf{c}_j|)}{|\mathbf{y} - \mathbf{c}_j|} \delta_{\lambda'}^{\lambda} + \left(\frac{g'_j(|\mathbf{y} - \mathbf{c}_j|)}{|\mathbf{y} - \mathbf{c}_j|^2} - \frac{g_j(|\mathbf{y} - \mathbf{c}_j|)}{|\mathbf{y} - \mathbf{c}_j|^3} \right) (\mathbf{y} - \mathbf{c}_j)^{\lambda} (\mathbf{y} - \mathbf{c}_j)^{\lambda'}, \quad (2.14)$$

$$\mathbf{y} \in B_{b_j}(\mathbf{c}_j), \quad 1 \leq j \leq N,$$

$$A_{\lambda'}^{\lambda} = \delta_{\lambda'}^{\lambda}, \quad \mathbf{y} \in \mathbb{R}_0^3 \setminus \bigcup_{j=1}^N B_{b_j}(\mathbf{c}_j).$$

The determinant is equal to

$$\Delta(\mathbf{y}) = g'_j(|\mathbf{y} - \mathbf{c}_j|) \left(\frac{g_j(|\mathbf{y} - \mathbf{c}_j|)}{|\mathbf{y} - \mathbf{c}_j|} \right)^2, \quad \mathbf{y} \in B_{b_j}(\mathbf{c}_j), \quad 1 \leq j \leq N, \quad (2.15)$$

$$\Delta(\mathbf{y}) = 1, \quad \mathbf{y} \in \mathbb{R}_0^3 \setminus \bigcup_{j=1}^N B_{b_j}(\mathbf{c}_j).$$

Note that Δ diverges at the boundary of K .

We take here the *material interpretation* and we consider our transformation as a bijection between two different spaces, Ω_0 and Ω . However, our transformation can be considered, as well, as a change of coordinates in Ω_0 . These two points of view are mathematically equivalent. This means that under our transformation Maxwell equations in Ω_0 and in Ω have the same invariance that they have under change of coordinates in 3-space (see, for example, [23]). Let us denote by $\mathbf{E}_0, \mathbf{H}_0, \mathbf{B}_0, \mathbf{D}_0, \varepsilon_0^{\lambda\nu}, \mu_0^{\lambda\nu}$, respectively, the electric and magnetic fields, the magnetic induction, the electric displacement, and the permittivity and permeability of Ω_0 . $\varepsilon_0^{\lambda\nu}, \mu_0^{\lambda\nu}$ are positive Hermitian matrices that are constant in Ω_0 . The electric field is a covariant vector that transforms as

$$E_{\lambda}(\mathbf{x}) = A_{\lambda}^{\lambda'}(\mathbf{y}) E_{0,\lambda'}(\mathbf{y}). \quad (2.16)$$

The magnetic field \mathbf{H} is a covariant pseudo-vector, but as we only consider space transformations with positive determinant, it also transforms as in (2.16). The magnetic induction \mathbf{B} and the electric displacement \mathbf{D} are contravariant vector densities of weight one that transform as

$$B^\lambda(\mathbf{x}) = (\Delta(\mathbf{y}))^{-1} A_{\lambda'}^\lambda(\mathbf{y}) B_0^{\lambda'}(\mathbf{y}), \quad (2.17)$$

with the same transformation for \mathbf{D} . The permittivity and permeability are contravariant tensor densities of weight one that transform as

$$\varepsilon^{\lambda\nu}(\mathbf{x}) = (\Delta(\mathbf{y}))^{-1} A_{\lambda'}^\lambda(\mathbf{y}) A_{\nu'}^\nu(\mathbf{y}) \varepsilon_0^{\lambda'\nu'}, \quad (2.18)$$

with the same transformation for $\mu^{\lambda\nu}$. Maxwell equations (2.1)–(2.4) are the same as in both spaces Ω and Ω_0 . Let us denote by $\varepsilon_{\lambda\nu}$, $\mu_{\lambda\nu}$, $\varepsilon_{0\lambda\nu}$, $\mu_{0\lambda\nu}$, respectively, the inverses of the corresponding permittivity and permeability. They are covariant tensor densities of weight minus one that transform as

$$\varepsilon_{\lambda\nu}(\mathbf{x}) = \Delta(\mathbf{y}) A_{\lambda'}^\lambda(\mathbf{y}) A_{\nu'}^\nu(\mathbf{y}) \varepsilon_{0\lambda'\nu'}, \quad \mu_{\lambda\nu}(\mathbf{x}) = \Delta(\mathbf{y}) A_{\lambda'}^\lambda(\mathbf{y}) A_{\nu'}^\nu(\mathbf{y}) \mu_{0\lambda'\nu'}. \quad (2.19)$$

We have that

$$\det \varepsilon^{\lambda\nu} = \Delta^{-1} \det \varepsilon_0^{\lambda\nu}, \quad \det \mu^{\lambda\nu} = \Delta^{-1} \det \mu_0^{\lambda\nu}, \quad (2.20)$$

$$\det \varepsilon_{\lambda\nu} = \Delta \det \varepsilon_{0\lambda\nu}, \quad \det \mu_{\lambda\nu} = \Delta \det \mu_{0\lambda\nu}. \quad (2.21)$$

The matrices $\varepsilon^{\lambda\nu}$, $\mu^{\lambda\nu}$ are degenerate at ∂K and the matrices $\varepsilon_{\lambda\nu}$, $\mu_{\lambda\nu}$ are singular at ∂K .

As the $\varepsilon^{\lambda\nu}$ and $\mu^{\lambda\nu}$ are degenerate at the boundary of the cloaked object K , we have to make precise what do we mean by a solution to Maxwell equations in neighborhood of ∂K . In other words, we have to specify the *cloaking boundary conditions* that the solutions have to satisfy on the outside and inside of ∂K . We solved this problem in [4, 5] by requiring that the fundamental principle of energy conservation be satisfied. That is to say, we obtained the appropriate boundary conditions by requiring that the solutions conserve energy. Note that as our media are loss-less energy has to be conserved. Furthermore, any energy loss of the incoming waves could *a priori* be detected, and, in consequence, energy conservation is essential for cloaking purposes. We briefly review the results of [4, 5].

We first consider the problem in Ω . We write Maxwell equations in Schrödinger form. For this purpose we denote by ε and μ , respectively, the matrices with entries $\varepsilon_{\lambda\nu}$ and $\mu_{\lambda\nu}$. Recall that $(\nabla \times \mathbf{E})^\lambda = s^{\lambda\nu\rho} \frac{\partial}{\partial x_\nu} E_\rho$, where $s^{\lambda\nu\rho}$ is the permutation contravariant pseudo-density of weight -1 (see section 6 of chapter II of [23], where a different notation is used).

We define the following formal differential operator:

$$a_\Omega \begin{pmatrix} \mathbf{E} \\ \mathbf{H} \end{pmatrix} = \mathbf{i} \begin{pmatrix} \varepsilon \nabla \times \mathbf{H} \\ -\mu \nabla \times \mathbf{E} \end{pmatrix}. \quad (2.22)$$

Here, as usual, we denote $\varepsilon \nabla \times \mathbf{H} := \varepsilon_{\lambda\nu} (\nabla \times \mathbf{H})^\nu$ and $\mu \nabla \times \mathbf{E} = \mu_{\lambda\nu} (\nabla \times \mathbf{E})^\nu$.

Equation (2.1) is equivalent to

$$\mathbf{i} \frac{\partial}{\partial t} \begin{pmatrix} \mathbf{E} \\ \mathbf{H} \end{pmatrix} = a_\Omega \begin{pmatrix} \mathbf{E} \\ \mathbf{H} \end{pmatrix}, \quad (2.23)$$

and equation (2.3) is equivalent to

$$-\omega \begin{pmatrix} \mathbf{E} \\ \mathbf{H} \end{pmatrix} = a_\Omega \begin{pmatrix} \mathbf{E} \\ \mathbf{H} \end{pmatrix}. \quad (2.24)$$

Note that since the matrices ε , μ are singular at $\partial\Omega$ the operator a_Ω has coefficients that are singular at $\partial\Omega$. This is the reason why we have to be careful when defining the solutions.

It is necessary to define equation (2.22) in an appropriate linear subspace of the Hilbert space of all finite-energy fields in Ω . We designate by $\mathcal{H}_{\Omega E}$ the Hilbert space of all measurable, \mathbb{C}^3 -valued functions defined on Ω that are square integrable with the weight $\varepsilon^{\lambda\nu}$ and the scalar product

$$(\mathbf{E}^{(1)}, \mathbf{E}^{(2)})_{\Omega E} := \int_{\Omega} E_{\lambda}^{(1)} \varepsilon^{\lambda\nu} \overline{E_{\nu}^{(2)}} \, d\mathbf{x}^3. \quad (2.25)$$

Moreover, we denote by $\mathcal{H}_{\Omega H}$ the Hilbert space of all measurable, \mathbb{C}^3 -valued functions defined on Ω that are square integrable with the weight $\mu^{\lambda\nu}$ and the scalar product

$$(\mathbf{H}^{(1)}, \mathbf{H}^{(2)})_{\Omega H} := \int_{\Omega} H_{\lambda}^{(1)} \mu^{\lambda\nu} \overline{H_{\nu}^{(2)}} \, d\mathbf{x}^3. \quad (2.26)$$

The Hilbert space of finite-energy fields in Ω is the direct sum

$$\mathcal{H}_{\Omega} := \mathcal{H}_{\Omega E} \oplus \mathcal{H}_{\Omega H}. \quad (2.27)$$

We first define a_{Ω} in a nice set of functions where it makes sense, which we take as $\mathbf{C}_0^1(\Omega)$. In physical terms this means that we start with the minimal assumption that Maxwell's equations are satisfied in classical sense away from the boundary of Ω . a_{Ω} with domain $D(a_{\Omega}) := \mathbf{C}_0^1(\Omega)$ is a symmetric operator in \mathcal{H}_{Ω} , i.e. $a_{\Omega} \subset a_{\Omega}^*$. To construct a unitary dynamics that preserves energy we have to analyze the self-adjoint extensions of a_{Ω} , what in physical terms means that we have to make precise in what sense Maxwell's equations are solved up to $\partial\Omega$. In other words, to construct finite-energy solutions of (2.23), with constant energy we have to demand that the initial finite-energy fields $(\mathbf{E}(0), \mathbf{H}(0))^T$ belong to the domain of one of the self-adjoint extensions of a_{Ω} . The key issue is that a_{Ω} has only one self-adjoint extension (i.e. it is essentially self-adjoint) that we denote by A_{Ω} . Moreover, A_{Ω} is unitarily equivalent to the free Maxwell propagator, A_0 , in Ω_0 . The unitary equivalence is generated by (2.16) and by the same transformation for the magnetic field. This means that there is only one dynamics in Ω that preserves energy, and that this dynamics is generated by A_{Ω} . As A_{Ω} and A_0 are unitarily equivalent, the dynamics that they generate are physically equivalent, and this is the deep reason, from the point of view of fundamental physics, why there is perfect cloaking of passive and active devices.

Solutions to (2.3), (2.4) in general do not have finite energy because they do not have enough decay at infinity to be square integrable over all Ω . Then, we only require that they are of locally finite energy in the sense that the electric and the magnetic fields are square integrable over every bounded subset of Ω , respectively, with the weights $\varepsilon^{\lambda\nu}$ and $\mu^{\lambda\mu}$. Moreover, in order that the problem (2.3), (2.4) is well-posed—in the sense that it is self-adjoint—the solutions with locally finite energy have to be locally in the domain of the only self-adjoint extension of a_{Ω} , that is to say, they have to be in the domain of A_{Ω} when multiplied by any continuously differentiable function with support in a bounded subset of $\overline{\Omega}$.

On the basis of these considerations we proved in [4, 5] that the solutions with locally finite energy in Ω are solutions in distribution sense to (2.3), (2.4) that satisfy

$$\int_O E_{\lambda} \varepsilon^{\lambda\nu} \overline{E_{\nu}} \, d\mathbf{x}^3 + \int_O H_{\lambda} \mu^{\lambda\nu} \overline{H_{\nu}} \, d\mathbf{x}^3 < \infty, \quad (2.28)$$

for every bounded set $O \subset \Omega$. Moreover, they have to satisfy the *cloaking boundary condition*

$$\mathbf{E} \times \mathbf{n} = 0, \quad \mathbf{H} \times \mathbf{n} = 0, \quad \text{in } \partial\Omega = \partial K_+, \quad (2.29)$$

where ∂K_+ is the outside of the boundary of the cloaked object and \mathbf{n} is the normal vector to ∂K_+ .

Note that as A_{Ω} is the only self-adjoint extension of a_{Ω} , this is the only possible self-adjoint boundary condition on ∂K_+ . It is self-adjoint because the matrices ε, μ are singular at

∂K_+ . Hence, cloaking as boundary value problem consists of finding a solution to (2.3), (2.4) in Ω with locally finite energy that satisfies the *cloaking boundary condition* given in (2.29).

Let us now consider the propagation of electromagnetic waves inside the cloaked object. We assume that in each K_j the permittivity and the permeability are given by $\varepsilon_j^{\lambda\nu}, \mu_j^{\lambda\nu}$, with inverses $\varepsilon_{j\lambda\nu}, \mu_{j\lambda\nu}$ and where ε_j, μ_j are the matrices with entries $\varepsilon_{j\lambda\nu}, \mu_{j\lambda\nu}$. Furthermore, we assume that $0 < \varepsilon_j^{\lambda\nu}(\mathbf{x}), \mu_j^{\lambda\nu}(\mathbf{x}) \leq C, \mathbf{x} \in K_j$ and that for any compact set Q contained in the interior of K_j there is a positive constant C_Q such that $\det \varepsilon_j^{\lambda\nu}(\mathbf{x}) > C_Q, \det \mu_j^{\lambda\nu}(\mathbf{x}) > C_Q, \mathbf{x} \in Q, j = 1, 2, \dots, N$. In other words, we only allow for possible singularities of ε_j, μ_j on the boundary of K_j .

We designate by \mathcal{H}_{jE} the Hilbert space of all measurable, \mathbb{C}^3 -valued functions defined on K_j that are square integrable with the weight $\varepsilon_j^{\lambda\nu}$ and the scalar product

$$(\mathbf{E}_j^{(1)}, \mathbf{E}_j^{(2)})_{jE} := \int_{K_j} E_{j\lambda}^{(1)} \varepsilon_j^{\lambda\nu} \overline{E_{j\nu}^{(2)}} \, d\mathbf{x}^3. \quad (2.30)$$

Similarly, we denote by \mathcal{H}_{jH} the Hilbert space of all measurable, \mathbb{C}^3 -valued functions defined on K_j that are square integrable with the weight $\mu_j^{\lambda\nu}$ and the scalar product

$$(\mathbf{H}_j^{(1)}, \mathbf{H}_j^{(2)})_{jH} := \int_{K_j} H_{j\lambda}^{(1)} \mu_j^{\lambda\nu} \overline{H_{j\nu}^{(2)}} \, d\mathbf{x}^3. \quad (2.31)$$

The Hilbert space of finite-energy fields in K_j is the direct sum

$$\mathcal{H}_j := \mathcal{H}_{jE} \oplus \mathcal{H}_{jH}, \quad (2.32)$$

and the Hilbert space of finite-energy fields in the cloaked object K is the direct sum,

$$\mathcal{H}_K := \bigoplus_{j=1}^N \mathcal{H}_j.$$

The complete Hilbert space of finite-energy fields including the cloaked object is

$$\mathcal{H} := \mathcal{H}_\Omega \oplus \mathcal{H}_K. \quad (2.33)$$

We now write (2.1) as a Schrödinger equation in each K_j as before. We define the following formal differential operator:

$$a_j \begin{pmatrix} \mathbf{E}_j \\ \mathbf{H}_j \end{pmatrix} = i \begin{pmatrix} \varepsilon_j \nabla \times \mathbf{H}_j \\ -\mu_j \nabla \times \mathbf{E}_j \end{pmatrix}. \quad (2.34)$$

Equation (2.1) in K_j is equivalent to

$$i \frac{\partial}{\partial t} \begin{pmatrix} \mathbf{E}_j \\ \mathbf{H}_j \end{pmatrix} = a_j \begin{pmatrix} \mathbf{E}_j \\ \mathbf{H}_j \end{pmatrix}. \quad (2.35)$$

Let us denote the interior of K_j by $\overset{\circ}{K}_j := K_j \setminus \partial K_j$. Then, a_j with domain $C_0^1(\overset{\circ}{K}_j)$ is a symmetric operator in \mathcal{H}_j . We denote

$$a := a_\Omega \oplus a_K, \quad \text{where } a_K := \bigoplus_{j=1}^N a_j, \quad (2.36)$$

with domain

$$D(a) := \left\{ \begin{pmatrix} \mathbf{E}_\Omega \\ \mathbf{H}_\Omega \end{pmatrix} \oplus_{j=1}^N \begin{pmatrix} \mathbf{E}_j \\ \mathbf{H}_j \end{pmatrix} \in C_0^1(\Omega) \oplus_{j=1}^N C_0^1(\overset{\circ}{K}_j) \right\}. \quad (2.37)$$

The operator a is symmetric in \mathcal{H} . The possible unitary dynamics that preserve energy for the whole system, including the cloaked object, K , are given by the self-adjoint extensions of a . We proved in [4, 5] that every self-adjoint extension, A , of a is the direct sum of A_Ω and of some self-adjoint extension, A_K , of a_K , i.e.,

$$A = A_\Omega \oplus A_K. \quad (2.38)$$

This result implies that the cloaked object K and the exterior Ω are completely decoupled. That electromagnetic waves outside K cannot go inside and vice versa, that waves inside cannot propagate outside and that there is perfect cloaking of passive and active devices. Choosing a particular self-adjoint extension A_K amounts to fixing a boundary condition in the inside of the boundary of the cloaked object, ∂K_- . The self-adjoint extension, or boundary condition, that nature will take depends on the properties of the media inside the cloaked object. Note that this does not mean that we have to put any physical surface, a lining, on the surface of the cloaked object to enforce any particular boundary condition on the inside, since this plays no role in the cloaking outside. It is, however, of importance to determine what the self-adjoint extension in K , or the interior boundary condition, has to be for specific cloaked objects. See [4, 5] for a detailed discussion of these issues.

The problem that we address in this paper is to determine what the boundary conditions in ∂K_- have to be in the particular case where the permittivity and the permeability in K are bounded and non-degenerate, i.e., when the matrices $\varepsilon_j^{\lambda\nu}, \mu_j^{\lambda\nu}$ are bounded above and below in K_j ,

$$0 < C_1 < \varepsilon_j^{\lambda\nu}, \quad \mu_j^{\lambda\nu} < C_2, \quad \mathbf{x} \in K_j, \quad j = 1, 2, \dots, N, \quad (2.39)$$

for some positive constants C_1, C_2 . This is clearly the most important case in the applications. It corresponds to a standard object that is cloaked with a metamaterial.

Let us consider the case of an active device with electric and magnetic currents in K . The Maxwell equations at frequency ω are

$$\nabla \times \mathbf{H} = i\omega \mathbf{D} + J, \quad (2.40)$$

$$\nabla \times \mathbf{E} = -i\omega \mathbf{B} - J_m, \quad (2.41)$$

where J and J_m are, respectively, the electric and the magnetic currents that we assume are different from zero only in K .

As we mentioned above, we have already proven in [4, 5] that energy conservation implies that the electromagnetic waves inside K cannot propagate outside and that, vice versa, the waves outside cannot go inside. The key issue here is that this is consistent with Maxwell equation (2.40) only if the normal component of the total current (i.e. the sum of the displacement current and the electric current) vanishes at ∂K_- , i.e., if

$$(i\omega \mathbf{D} + J) \cdot \mathbf{n}|_{\partial K_-} = 0, \quad (2.42)$$

where as usual by ∂K_- we mean that we approach the boundary from the inside. In a similar way, the consistency with Maxwell equation (2.41) implies that

$$(i\omega \mathbf{B} + J_m) \cdot \mathbf{n}|_{\partial K_-} = 0. \quad (2.43)$$

Note that we do not need to ask that (2.42), (2.43) hold at ∂K_+ because as we assume that J, J_m are identically zero outside K conditions (2.42), (2.43) in ∂K_+ follow from equations (2.14), (2.15), (2.17) and the same transformation equation for \mathbf{D} , since the solution in Ω is obtained applying the transformation formulae to a solution in \mathbb{R}_0^3 [4, 5]. Moreover, the boundary conditions (2.42), (2.43) on ∂K_- and Maxwell equations (2.40), (2.41) imply that

$$(\nabla \times \mathbf{H}) \cdot \mathbf{n}|_{\partial K_-} = 0, \quad (\nabla \times \mathbf{E}) \cdot \mathbf{n}|_{\partial K_-} = 0. \quad (2.44)$$

Hence, we have proven that the boundary conditions that we have to impose on the inside of the boundary of the cloaked object are (2.44), namely that the normal components of the curl of the electric and the magnetic fields have to vanish. The Maxwell propagator a_K with the boundary condition (2.44) has already been studied in the mathematical literature,

in particular in relation with Beltrami fields. As it was to be expected from our analysis, a_K with the boundary condition (2.44) is a self-adjoint operator. In other words, the boundary condition (2.44) defines the self-adjoint realization, A_K , of the Maxwell propagator in K that is imposed by energy conservation. For the proof of self-adjointness, as well as other issues, including the formulation of (2.44) in weak sense see [24–27], in particular, see page 158, theorem 2.1, corollary 2.1.1, page 164 and theorem 2.3 of [26]. Note that in our case the Neumann fields are zero. Remark that imposing that the normal components of D and B are zero at ∂K_- is not a self-adjoint boundary condition, i.e., it does not define a self-adjoint extension of Maxwell generator in K .

We have now a complete formulation of cloaking as a boundary value problem. It consists of finding a solution of Maxwell equations (2.40), (2.41) in distribution sense in $\mathbb{R}^3 \setminus \partial K$, with locally finite energy, i.e., they satisfy

$$\int_O E_\lambda \varepsilon^{\lambda\nu} \overline{E}_\nu \, d\mathbf{x}^3 + \int_O H_\lambda \mu^{\lambda\nu} \overline{H}_\nu \, d\mathbf{x}^3 < \infty, \quad (2.45)$$

where O is any bounded subset of \mathbb{R}^3 . Moreover, they have to satisfy the *cloaking boundary conditions*

$$\mathbf{E} \times \mathbf{n} = 0, \quad \mathbf{H} \times \mathbf{n} = 0, \quad \text{at } \partial\Omega = \partial K_+, \quad (2.46)$$

and

$$(\nabla \times \mathbf{E}) \cdot \mathbf{n} = 0, \quad (\nabla \times \mathbf{H}) \cdot \mathbf{n} = 0, \quad \text{at } \partial K_-. \quad (2.47)$$

We have derived the boundary conditions (2.46), (2.47) by requiring that the solutions to the fixed frequency Maxwell equations (2.40), (2.41) are (locally) in the domain of the appropriate self-adjoint extension (2.38) of Maxwell generator (2.36). Note that when we define the self-adjoint operator A_K we have to require that all functions on its domain satisfy the boundary conditions (2.47), not just the solutions to the fixed frequency Maxwell equations. In fact, the choice of the self-adjoint Maxwell generator A_K has implications that go well beyond the formulation of cloaking as a boundary value problem. For example, it determines the time evolution of finite-energy wave packets in the time domain (see [4, 5] for this issue).

3. The case of a radial source at the boundary of K

In this section, we illustrate our method by considering an active device given by a radial electric current at the boundary of K . The case of a magnetic current at the boundary of K follows in the same way. We assume that K is isotropic and spherically stratified, i.e., that the permittivity and the permeability depend only on $|\mathbf{x}|$. The case where K is isotropic and homogeneous, and with an electric dipole contained in the interior of K was already considered in [14]. We verify in this particular case, by an explicit computation, that our *cloaking boundary conditions* are satisfied and that cloaking of active devices holds, even if the current is at the boundary of the cloaked object, as we have proven in section 2 in the general case where there is no explicit solution.

We assume that we have only one spherical cloak, K , located at the origin, i.e., $N = 1$, $\mathbf{c}_1 = 0$, that K is isotropic with permittivity and permeability, ε_1, μ_1 , that are bounded, that they have a positive lower bound and that they depend only on $r := |\mathbf{x}|$. For simplicity we take a first-order transformation with $g(\rho) = \frac{b-a}{b}\rho + a$, $0 < a < b$. In the cloaking layer the permittivity and the permeability tensors are given by [1, 16]

$$\varepsilon^{\lambda\nu} = \varepsilon_r \hat{\mathbf{x}}^\lambda \hat{\mathbf{x}}^\nu + \varepsilon_t \hat{\theta}^\lambda \hat{\theta}^\nu + \varepsilon_l \hat{\phi}^\lambda \hat{\phi}^\nu, \quad \mu^{\lambda\nu} = \mu_r \hat{\mathbf{x}}^\lambda \hat{\mathbf{x}}^\nu + \mu_t \hat{\theta}^\lambda \hat{\theta}^\nu + \mu_l \hat{\phi}^\lambda \hat{\phi}^\nu, \quad a < |\mathbf{x}| < b, \quad (3.1)$$

where $\hat{\theta}$, $\hat{\varphi}$ are unit tangent vectors, respectively, to the coordinate lines r , φ constant and r , θ constant, in spherical coordinates r , θ , φ . Moreover,

$$\varepsilon_t/\varepsilon_0 = \mu_t/\mu_0 = b/(b - a), \quad \varepsilon_r/\varepsilon_t = \mu_r/\mu_t = (r - a)^2/r^2, \quad a < r < b. \quad (3.2)$$

We assume that for $r > b$ the medium is homogeneous and isotropic with permittivity and permeability, ε_0 , μ_0 .

The expression of the transverse electric (TE) and transverse magnetic (TM) fields in terms of potentials given in section 8.6 of [8] remains true in our case (remark the ε_t and μ_t are constant in the cloaking layer) (see also [13, 14]). TE and TM fields decouple, and since we have a radial electric current we only consider TM modes. Assuming that $J = J_r(r)\hat{x}$ and that $J_m = 0$, the TM fields are given by the potential as follows [8]:

$$\begin{aligned} \mathbf{E}_t &= \frac{-i}{\omega\varepsilon_t r} \text{grad}_{\theta\varphi} \frac{d}{dr} I, \\ \mathbf{H}_t &= \frac{1}{r} (\text{grad}_{\theta\varphi} I \times \hat{\mathbf{x}}), \\ \mathbf{E}_r &= -\frac{1}{i\omega\varepsilon_r r^2} \Delta_{\theta\varphi} I - \frac{1}{i\omega\varepsilon_r} J_r. \end{aligned} \quad (3.3)$$

We expand the potential in spherical harmonics

$$I = \sum_{mn} I(r)_{mn} Y_n^m(\theta, \varphi),$$

and we assume that the radial current has the following expansion:

$$J_r = \sum_{mn} J_{mn} \delta(r - a) Y_n^m(\theta, \varphi),$$

for some constants J_{mn} . For example, for an electric dipole located at $(0, 0, a)$ we have that $J_{mn} = -\frac{i\omega P_e}{2\pi a^2} \sqrt{\frac{2n+1}{4\pi}} \delta_{m,0}$ [8].

We now set the inner boundary at $a + \delta$, for small $\delta > 0$, i.e., we assume that the permittivity and permeability are equal to ε_1 , μ_1 for $r < a + \delta$, that they are given by (3.1), (3.2) for $a + \delta < r < b$ and by ε_0 , μ_0 for $r > b$. We compute the solution, and then we take the limit as δ tends to zero [19].

For $0 < r < a + \delta$, the potential $I_{a,mn}$ created by the source satisfies the following equation [8], where we denote $\frac{d}{dr}$ by $'$,

$$I''_{a,mn} - \frac{1}{\varepsilon_1} \varepsilon'_1 I'_{a,mn} + \left[\omega^2 \varepsilon_1 \mu_1 - \frac{n(n+1)}{r^2} \right] I_{a,mn} = -J_{mn} \delta(r - a). \quad (3.4)$$

Let v_n and w_n to be independent solutions of the homogeneous equation with v_n regular at zero. For example, if $\varepsilon_1(r)$, $\mu_1(r)$ are piecewise constant, v_n can be taken as a Ricatti–Bessel function of the first kind and w_n as a Ricatti–Bessel function of the second or third kind [9] in each layer where ε_1 , μ_1 are constant. Hence, the solution to (3.4) is given by

$$I_{a,mn}(r) = \frac{J_{mn}}{v'_n(a)w_n(a) - v_n(a)w'_n(a)} \begin{cases} w_n(a)v_n(r), & 0 < r < a, \\ v_n(a)w_n(r), & a < r < a + \delta. \end{cases} \quad (3.5)$$

We suppose that $v_n(a)$ and $w_n(a)$ are different from zero. We assume that there is also a reflected wave. Then, the total potential is given by

$$I_{mn} = I_{a,mn}(r) + R_{mn}v_n(r), \quad 0 < r < a + \delta, \quad (3.6)$$

where the R_{mn} are the reflection coefficients. In the cloaking layer the potential satisfies the equation (remark that ε_t, μ_t are constant)

$$I''_{mn} + \left[\omega^2 \varepsilon_t \mu_t - \frac{\varepsilon_t}{\varepsilon_r} \frac{n(n+1)}{r^2} \right] I_{mn} = 0, \quad a + \delta < r < b. \quad (3.7)$$

The solution is given by Ricatti–Bessel functions of the first and second kind,

$$I_{mn}(r) = c_{mn} \psi_n(k_t(r-a)) + d_{mn} \chi_n(k_t(r-a)), \quad k_t := \omega \sqrt{\varepsilon_t \mu_t}, \quad a + \delta < r < b. \quad (3.8)$$

Outside of the cloaking layer the potential satisfies equation (3.7) with $\varepsilon_t = \varepsilon_r = \varepsilon_0$ and $\mu_t = \mu_0$. The solution is an outgoing wave,

$$I_{mn}(r) = T_{mn} \zeta_n(k_0 r), \quad k_0 := \omega \sqrt{\varepsilon_0 \mu_0}, \quad b < r, \quad (3.9)$$

where ζ is a Ricatti–Bessel function of the third kind.

Requiring that the tangential components of the electric and the magnetic fields are continuous at $a + \delta$ and b we obtain the following equations:

$$\frac{1}{\varepsilon_1(a+\delta)} (I'_{a,mn}(a+\delta) + R_{mn} v'_n(a+\delta)) = \frac{k_t}{\varepsilon_t} (c_{mn} \psi'_n(k_t \delta) + d_{mn} \chi'_n(k_t \delta)), \quad (3.10)$$

$$I_{a,mn}(a+\delta) + R_{mn} v_n(a+\delta) = c_{mn} \psi_n(k_t \delta) + d_{mn} \chi_n(k_t \delta), \quad (3.11)$$

$$\frac{k_t}{\varepsilon_t} (c_{mn} \psi'_n(k_t(b-a)) + d_{mn} \chi'_n(k_t(b-a))) = \frac{k_0}{\varepsilon_0} T_{mn} \zeta'_n(k_0 b), \quad (3.12)$$

$$c_{mn} \psi_n(k_t(b-a)) + d_{mn} \chi_n(k_t(b-a)) = T_{mn} \zeta_n(k_0 b). \quad (3.13)$$

Let us denote

$$\alpha := \left(\frac{\mu_t \varepsilon_0}{\mu_0 \varepsilon_t} \right)^{1/2}, \quad (3.14)$$

$$\beta_1 := \zeta'_n(k_0 b) \chi_n(k_t(b-a)) - \alpha \zeta_n(k_0 b) \chi'_n(k_t(b-a)), \quad (3.15)$$

$$\beta_2 := \zeta'_n(k_0 b) \psi_n(k_t(b-a)) - \alpha \zeta_n(k_0 b) \psi'_n(k_t(b-a)), \quad \beta_3 := -\frac{\beta_1}{\beta_2}, \quad (3.16)$$

$$\beta_4 := v'_n(a+\delta) (\beta_3 \psi_n(k_t \delta) + \chi_n(k_t \delta)) - \frac{k_t \varepsilon_1 (a+\delta)}{\varepsilon_t} v_n(a+\delta) (\beta_3 \psi'_n(k_t \delta) + \chi'_n(k_t \delta)). \quad (3.17)$$

Solving (3.10)–(3.13) in the generic case where $\beta_2 \neq 0$ we obtain that

$$R_{mn} = \frac{1}{v_n(a+\delta)} ((\beta_3 \psi_n(k_t \delta) + \chi_n(k_t \delta)) d_{mn} - I_{a,mn}(a+\delta)), \quad (3.18)$$

$$c_{mn} = \beta_3 d_{mn}, \quad (3.19)$$

$$d_{mn} = J_{mn} v_n(a) \frac{1}{\beta_4}, \quad (3.20)$$

$$T_{mn} = \frac{1}{\zeta_n(k_0 b)} (\beta_3 \psi_n(k_t(b-a)) + \chi_n(k_t(b-a))) d_{mn}. \quad (3.21)$$

Using the expansions of the spherical Bessel functions for small argument given in equations (10.1.2), (10.1.3) of [28] we prove that

$$d_{mn} = -J_{mn} \frac{v_n(a) \varepsilon_t}{k_t \varepsilon_1 (a+\delta) v_n(a+\delta) n(2n-1)!!} (k_t \delta)^{n+1} + O((k_t \delta)^{n+2}). \quad (3.22)$$

Using (3.22) we obtain the small δ expansions of R_{mn} , c_{mn} , T_{mn} , respectively, from (3.18), (3.19), (3.21),

$$R_{mn} = \frac{1}{v_n(a + \delta)} (-I_{a,mn}(a + \delta) + O(k_t \delta)), \quad (3.23)$$

$$c_{mn} = \beta_3 \left(-J_{mn} \frac{v_n(a) \varepsilon_t}{k_t \varepsilon_1(a + \delta) v_n(a + \delta) n(2n - 1)!!} (k_t \delta)^{n+1} + O((k_t \delta)^{n+2}) \right), \quad (3.24)$$

$$T_{mn} = -\frac{J_{mn} v_n(a) \varepsilon_t (\beta_3 \psi_n(k_t(b - a)) + \chi_n(k_t(b - a)))}{k_t \varepsilon_1(a + \delta) \zeta_n(k_0 b) v_n(a + \delta) n(2n - 1)!!} (k_t \delta)^{n+1} + O((k_t \delta)^{n+2}). \quad (3.25)$$

Moreover, by (3.6), (3.11), (3.22), (3.24),

$$I_{mn}(a + \delta) = J_{mn} \frac{v_n(a) \varepsilon_t}{v_n(a + \delta) n k_t \varepsilon_1(a + \delta)} k_t \delta + O((k_t \delta)^2). \quad (3.26)$$

It follows that when $\delta = 0$, $c_{mn} = d_{mn} = T_{mn} = 0$, and the electric and magnetic fields outside K are zero, as predicted by our theoretical results. Also, for $\delta = 0$, $I_{mn}(a) = 0$, and then by equation (3.3) the radial component of sum of the displacement current and the electric current vanishes at $r = a$. By Maxwell equation (2.40) the normal component of the curl of the magnetic field vanishes at $r = a$. The corresponding statement for the normal component of the curl of the electric field is trivial in this case by Maxwell equation (2.41) and as the magnetic field is transversal and the magnetic current is zero. Hence, the boundary conditions (2.44) are satisfied and cloaking holds even if the current is at the boundary of K , as predicted by our theoretical results.

4. Conclusions

The results of this paper and of [4, 5] give a complete rigorous mathematical analysis of point transformed electromagnetic invisibility cloaks. We solved the mathematical challenges posed by the fact that the permittivity and the permeability are degenerate at the boundary of the cloaked object K . In particular, we characterized all possible ways to define solutions of Maxwell equations that are compatible with energy conservation. This result was obtained by characterizing all possible boundary conditions at ∂K_{\pm} that are allowed by energy conservation. They correspond to all self-adjoint extensions of the Maxwell generator. As it turned out, all self-adjoint extensions are the direct sum of some self-adjoint extension inside K with a fixed self-adjoint extension outside K . This implies that the solutions inside and outside of K are completely decoupled from each other and that, in general, they are discontinuous at ∂K . We also proved that cloaking of passive and active devices always holds for all possible ways to define solutions that satisfy energy conservation, i.e., with self-adjoint boundary conditions. The boundary condition at ∂K_+ is always that the tangential components of both the electric and the magnetic field vanish. At ∂K_- the boundary condition can be any self-adjoint boundary condition for the Maxwell generator in K . The particular self-adjoint boundary condition that nature will take depends on the specific properties of the media inside K . In this paper, we solved the problem of determining the appropriate boundary conditions at ∂K_- in the case where the permittivity and the permeability inside K are bounded and have a positive lower bound. This corresponds to the situation where we have a standard object K —that could be anisotropic and inhomogeneous, but whose permittivity and permeability are neither singular nor degenerate—that is coated by a transformation medium that is degenerate at ∂K_+ . We also allow for passive and active devices in K . This is perhaps the more important case in the applications.

In this way, we have obtained a complete formulation of cloaking with passive and active devices as a boundary value problem. It consists of finding a solution of Maxwell equations (2.40), (2.41) at frequency ω , in distribution sense in $\mathbb{R}^3 \setminus \partial K$, with locally finite energy, i.e., they satisfy

$$\int_O E_\lambda \varepsilon^{\lambda\nu} \overline{E_\nu} \, dx^3 + \int_O H_\lambda \mu^{\lambda\nu} \overline{H_\nu} \, dx^3 < \infty,$$

where O is any bounded subset of \mathbb{R}^3 . Moreover, they have to satisfy the following *cloaking boundary conditions*:

$$\mathbf{E} \times \mathbf{n} = 0, \quad \mathbf{H} \times \mathbf{n} = 0, \quad \text{at } \partial K_+ \quad (4.1)$$

and

$$(\nabla \times \mathbf{E}) \cdot \mathbf{n} = 0, \quad (\nabla \times \mathbf{H}) \cdot \mathbf{n} = 0, \quad \text{at } \partial K_-. \quad (4.2)$$

In the case of one spherical cloaked object, that is isotropic and spherically stratified, and that has an active device given by a radial electric current at the boundary, we verified by an explicit computation that our *cloaking boundary conditions* are satisfied and that cloaking of active devices holds, even if the current is at the boundary of the cloaked object.

A novel aspect of our work is that we proved our results for transformation media that are obtained from general anisotropic media, i.e., that it is not necessary to transform from isotropic media. This means that it is possible to cloak objects that are contained inside general anisotropic materials, general crystals for example. For this purpose, we just have to take as the permittivity and the permeability of the general anisotropic medium before transformation those of the general anisotropic material, or the general crystal, which contains the object that we wish to cloak. The fact that it is possible to cloak objects inside general anisotropic media opens the way to other interesting potential applications, for example to guide electromagnetic waves under quite general circumstances.

The results above—that are proven for cloaks in exact transformation media (ideal cloaks)—set the stage for the rigorous study of the cloaks in the approximate transformation media that one has to consider in practical situations, what is one of the main open questions in this area. This issue can be understood as the problem of the stability of cloaking under the perturbation on the permittivity and the permeability given by the difference in the permittivity and the permeability between the exact and the approximate transformation media. Our formulation of cloaking as a self-adjoint problem shows that the important issue of stability of cloaking can be formulated as a problem in perturbation theory of the self-adjoint Maxwell generator. Perturbation theory of self-adjoint operators is a main stream topic in modern mathematical physics and there is a large body of results (see, for example, [29]). Our analysis opens the way to a rigorous study of the stability of cloaking along these lines.

For exact transformation media the boundary conditions (4.1), (4.2) are satisfied because they follow from energy conservation and there is no need to add any lining to impose them. However, our results suggest a method to enhance cloaking in the approximate transformation media that are used in practice. Namely, to coat ∂K by a material that imposes the boundary conditions (4.1), (4.2). As these boundary conditions have to be satisfied for exact transformation media, adding a lining that enforces them in the case of approximate transformation media will improve the performance of approximate cloaks.

Acknowledgments

I thank Rainer Picard for his information on his work on the self-adjoint realizations of the operator curl and of the Maxwell generator.

References

- [1] Pendry J B, Schurig D and Smith D R 2006 Controlling electromagnetic fields *Science* **312** 1780–2
- [2] Hendi A, Hen J and Leonhardt U 2006 Ambiguities in the scattering tomography for central potentials *Phys. Rev. Lett.* **97** 073902
- [3] Cai W, Chettiar U K, Kildishev A V, Milton G W and Shalaev V M 2007 Non-magnetic cloak without reflection arXiv:0707.3641
- [4] Weder R 2007 A rigorous time-domain analysis of full-wave electromagnetic cloaking (invisibility) arXiv:0704.0248
- [5] Weder R 2008 A rigorous analysis of high order electromagnetic invisibility cloaks *J. Phys. A: Math. Theor.* **41** 065207
- [6] Yan W, Yan M, Ruan Z and Qiu M 2008 Coordinate transformations make perfect invisibility cloaks with arbitrary shape *New J. Phys.* **10** 043040
- [7] Greenleaf A, Kurylev Y, Lassas M and Uhlmann G 2007 Full-wave invisibility of active devices at all frequencies *Commun. Math. Phys.* **275** 749–89
- [8] Van Bladel J 1985 *Electromagnetic Fields* (Washington: Hemisphere)
- [9] van de Hulst H C 1957 *Light Scattering by Small Particles* (New York: Dover)
- [10] Cai W, Chettiar U K, Kildishev A V and Shalaev V M 2007 Optical cloaking with metamaterials *Nature Photonics* **1** 224–6
- [11] Cummer S A, Popa B-I, Schurig D, Smith D R and Pendry J 2006 Full-wave simulation of electromagnetic cloaking structures *Phys. Rev. E* **74** 036621
- [12] Schurig D, Mock J J, Justice B J, Cummer S A, Pendry J B, Starr A F and Smith D R 2006 Metamaterial electromagnetic cloak at microwave frequencies *Science* **314** 977–80
- [13] Chen H, Wu B-I, Zhang B and Kong J A 2007 Electromagnetic wave interactions with a metamaterial cloak *Phys. Rev. Lett.* **99** 063903
- [14] Zhang B, Chen H, Wu B-I and Kong J A 2008 Extraordinary surface voltage effect in the invisibility cloak with an active device *Phys. Rev. Lett.* **100** 063904
- [15] Leonhardt U 2006 Optical conformal mapping *Science* **312** 1777–80
- [16] Schurig D, Pendry J B and Smith D R 2006 Calculation of material properties and ray tracing in transformation media *Opt. Exp.* **14** 9794–804
- [17] Smolyaninov I I, Hung Y J and Davis C C 2007 Electromagnetic cloaking in the visible frequency range arXiv:0709.2862
- [18] Greenleaf A, Kurylev Y, Lassas M and Uhlmann G 2007 Improvement of cylindrical cloaking with the SHS lining *Opt. Express* **15** 12717–34
- [19] Ruan Z, Yan M, Neff C W and Qiu M 2007 Ideal cylindrical cloak: Perfect but sensitive to tiny perturbations *Phys. Rev. Lett.* **99** 113903
- [20] Zhang B, Chen H, Wu B-I, Luo Y, Ran L and Kong J A 2007 Response of a cylindrical invisibility cloak to electromagnetic waves *Phys. Rev. B* **76** 121101
- [21] Greenleaf A, Lassas M and Uhlmann G 2003 Anisotropic conductivities that cannot be detected by EIT *Physiol. Meas.* **24** 413–9
- [22] Greenleaf A, Lassas M and Uhlmann G 2003 On nonuniqueness for Calderón’s inverse problem *Math. Res. Lett.* **10** 685–93
- [23] Post E J 1997 *Formal Structure of Electromagnetics General Covariance and Electromagnetics* (New York: Dover)
- [24] Picard R 1974 Zur Lösungstheorie der zeitunabhängiger Maxwell’schen Gleichungen mit der Randbedingung $\mathbf{n} \cdot \mathbf{B} = \mathbf{n} \cdot \mathbf{D} = 0$ in anisotropen, inhomogenen Medien *Manuscr. Math.* **13** 37–52
- [25] Picard R 1977 Ein Randwertproblem für die zeitunabhängigen Maxwell’schen Gleichungen mit der Randbedingung $\mathbf{n} \cdot \varepsilon \mathbf{E} = \mathbf{n} \cdot \mu \mathbf{H} = 0$ in beschränkten Gebieten beliebigen Zusammenhangs *Appl. Anal.* **6** 207–21
- [26] Picard R 1998 On a self-adjoint realization of curl and some of its applications *Ric. Mat.* **XLVII** 153–80
- [27] Filonov N 2000 Spectral analysis of the selfadjoint operator curl in a region of finite measure *St. Petersburg Math. J.* **11** 1085–95
- [28] Abramowitz M and Stegun I A 1970 *Handbook of Mathematical Functions* (New York: Dover)
- [29] Kato T 1995 *Perturbation Theory of Linear Operators* (Berlin: Springer)